

Self-sustained wormholes in modified dispersion relations

Remo Garattini*

*Università degli Studi di Bergamo, Facoltà di Ingegneria,
Viale Marconi 5, 24044 Dalmine (Bergamo) Italy and
and I.N.F.N. - sezione di Milano, Milan, Italy.*

Francisco S. N. Lobo†

*Centro de Astronomia e Astrofísica da Universidade de Lisboa, and
Campo Grande, Ed. C8 1749-016 Lisboa, Portugal*

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In this work, we consider the possibility that wormhole geometries are sustained by their own quantum fluctuations, in the context of modified dispersion relations. More specifically, the energy density of the graviton one-loop contribution to a classical energy in a wormhole background is considered as a self-consistent source for wormholes. In this semi-classical context, we consider specific choices for the Rainbow's functions and find solutions for wormhole geometries in the cis-planckian and trans-planckian regimes. In the former regime, the wormhole spacetimes are not asymptotically flat and need to be matched to an exterior vacuum solution. In the latter trans-planckian regime, we find that the quantum corrections are exponentially suppressed, which provide asymptotically flat wormhole geometries with a constant shape function, i.e., $b(r) = r_t$, where r_t is the wormhole throat. In addition to this analysis, we also fix the geometry by considering the behaviour of a specific shape function through a variational approach which imposes a local analysis to the problem at the wormhole throat. We further explore the respective parameter range of the Rainbow's functions, and find a good agreement with previous work.

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I. INTRODUCTION

Modified Dispersion Relations (MDR) are a distortion of the spacetime metric at energies comparable to the Planck energy. A general formalism, denoted as deformed or doubly special relativity [1] was developed in [2] in order to (i) preserve the relativity of inertial frames, (ii) maintain the Planck energy invariant and (iii) impose that in the limit $E/E_P \rightarrow 0$ the speed of a massless particle tends to a universal constant, c , which is the same for all inertial observers. In particular, in curved spacetime the formalism imposes that the relationship between the energy and momentum of a massive particle m in special relativity are modified at the Planck scale [1, 2], i.e., $E^2 g_1^2(E/E_P) - p^2 g_2^2(E/E_P) = m^2$, where the two unknown functions $g_1(E/E_P)$ and $g_2(E/E_P)$, denoted as the *Rainbow's functions*, have the following property

$$\lim_{E/E_P \rightarrow 0} g_1(E/E_P) = 1 \quad \text{and} \quad \lim_{E/E_P \rightarrow 0} g_2(E/E_P) = 1. \quad (1)$$

One interesting aspect of Rainbow's functions is the possibility of regularizing the divergent behaviour of some field quantities such as the energy density. This has been done in Ref. [3] with the following choice for the Rainbow's functions

$$g_1(E/E_P) = \sum_{i=0}^n \beta_i \frac{E^i}{E_P^i} \exp\left(-\alpha \frac{E^2}{E_P^2}\right), \quad g_2(E/E_P) = 1; \quad \alpha > 0, \beta_i \in \mathbb{R}. \quad (2)$$

More specifically, in Ref. [3], motivated by the promising results obtained in the application of Gravity's Rainbow to black hole entropy [4], it was shown that for an appropriate choice of the functions $g_1(E/E_P)$ and $g_2(E/E_P)$, the UV divergences of the Zero Point Energy disappear. It is interesting to note that every choice for the Rainbow's functions is restricted by convergence criteria, namely a pure polynomial cannot be used without the reintroduction of a regularization and a renormalization process.

*Electronic address: Remo.Garattini@unibg.it

†Electronic address: flobo@cii.fc.ul.pt

In fact, MDR have been considered in a wide variety of contexts, such as in cosmology [5] and black hole physics [6]. For instance, the *rainbow* version of the Schwarzschild line element is given by [2]

$$ds^2 = - \left(1 - \frac{2MG(0)}{r} \right) \frac{dt^2}{g_1^2(E/E_P)} + \frac{d\tilde{r}^2}{\left(1 - \frac{2MG(0)}{r} \right) g_2^2(E/E_P)} + \frac{\tilde{r}^2}{g_2^2(E/E_P)} (d\theta^2 + \sin^2 \theta d\phi^2) . \quad (3)$$

In particular, note that the position of the horizon, in fixed and energy independent coordinates, is at the usual fixed coordinate $2MG(0)$. However, it was shown that the area of the horizon is energy dependent [2].

In this paper, we use the scheme given by Rainbow's function(2), in order to verify if the MDR provide interesting solutions for wormhole geometries. The latter are hypothetical tunnels in spacetime [7, 8] and have been studied in a plethora of environments such as dark energy models [9], modified theories of gravity [10], observational signatures using thin accretion disks [11], and in the semi-classical regime [12–14], amongst many other contexts. In the semi-classical regime, the possibility that these wormholes are sustained by their own quantum fluctuations was considered in [13, 14]. The graviton one loop contribution to a classical energy in a wormhole background was taken into account, and a variational approach with Gaussian trial wave functionals was used. A zeta function regularization was involved to handle the divergences and a renormalization procedure was introduced. Thus, the finite one loop energy was considered as a *self-consistent* source for traversable wormholes. The aim of this work is to consider the possibility that wormhole geometries be sustained by their own quantum fluctuations, in the context of modified dispersion relations.

This paper is organized in the following manner: In Section II we briefly outline the formalism of the classical term in Rainbow's Gravity and the one loop energy in a spherically symmetric background. In Section III, we consider specific Rainbow's functions and find solutions of self-sustained wormhole geometries in the context of MDR. In Section IV, we fix the geometry by considering the behaviour of a specific metric function $b(r)$, through a variational approach, at the wormhole throat, r_t . In Section V, we conclude.

II. THE CLASSICAL TERM IN RAINBOW'S GRAVITY AND THE ONE LOOP ENERGY IN A SPHERICALLY SYMMETRIC BACKGROUND

Consider a wormhole spacetime described by the following line element

$$ds^2 = -N^2(r) \frac{dt^2}{g_1^2(E/E_P)} + \frac{dr^2}{\left(1 - \frac{b(r)}{r} \right) g_2^2(E/E_P)} + \frac{r^2}{g_2^2(E/E_P)} (d\theta^2 + \sin^2 \theta d\phi^2) , \quad (4)$$

where $N(r)$ is the lapse function and $b(r)$ is denoted the shape function, as can be shown by embedding diagrams, it determines the shape of the wormhole [7]. For simplicity, we consider $N(r) = 1$ throughout this work. A fundamental property of traversable wormholes is the flaring out condition of the throat, given by $(b - b')/b^2 > 0$ [7]. Note that the condition $1 - b/r > 0$ is also imposed. To be a wormhole solution the following conditions need to be satisfied at the throat: $b(r_t) = r_t$ and $b'(r_t) < 1$; the latter follows from the flaring out condition. Asymptotic flatness imposes $b(r)/r \rightarrow 0$ as $r \rightarrow +\infty$. However, one may also construct solutions by matching the interior solution to an exterior vacuum spacetime, at a junction interface, much in the spirit of [15].

In the analysis outlined below, we consider the graviton one loop contribution to a classical energy in a wormhole background [13], where the classical energy is given by

$$H_{\Sigma}^{(0)} = \int_{\Sigma} d^3x \mathcal{H}^{(0)} = -\frac{1}{16\pi G} \int_{\Sigma} d^3x \sqrt{g} R = -\frac{1}{2G} \int_{r_0}^{\infty} \frac{dr r^2}{\sqrt{1 - b(r)/r}} \frac{b'(r)}{r^2 g_2(E/E_P)} , \quad (5)$$

and the background field super-hamiltonian, $\mathcal{H}^{(0)}$, is integrated on a constant time hypersurface. Note that the graviton one loop contribution to a classical energy contribution is evaluated through a variational approach with Gaussian trial wave functionals, and the divergences are treated with a zeta function regularization. Using a renormalization procedure, the finite one loop energy was considered a self-consistent source for a traversable wormhole (we refer the reader to [13] for details).

In the following, we consider $g_{ij} = \bar{g}_{ij} + h_{ij}$, where h_{ij} is the quantum fluctuation around the background metric \bar{g}_{ij} . We shall also take into account the total regularized one loop energy given by

$$E^{TT} = -\frac{1}{2} \sum_{\tau} \frac{g_1(E/E_P)}{g_2^2(E/E_P)} \left[\sqrt{E_1^2(\tau)} + \sqrt{E_2^2(\tau)} \right] , \quad (6)$$

where $E_i^2(\tau) > 0$, and E_i are the eigenvalues of

$$\left(\hat{\Delta}_L^m h^\perp\right)_{ij} = \frac{E^2}{g_2^2(E/E_P)} h_{ij}^\perp. \quad (7)$$

h^\perp is the traceless-transverse component of the perturbation [16, 17] and

$$\left(\hat{\Delta}_L^m h^\perp\right)_{ij} = (\Delta_L h^\perp)_{ij} - 4R_i^k h_{kj}^\perp + {}^3R h_{ij}^\perp, \quad (8)$$

where Δ_L is the Lichnerowicz operator defined by

$$(\Delta_L h)_{ij} = \Delta h_{ij} - 2R_{ikjl} h^{kl} + R_{ik} h_j^k + R_{jk} h_i^k, \quad (9)$$

with $\Delta = -\nabla^a \nabla_a$. Rather than present all the intricate details here, we refer the reader to Ref. [3] for a detailed discussion on these issues.

Using the Regge and Wheeler representation [18], the eigenvalue equation (7) can be reduced to

$$\left[-\frac{d^2}{dx^2} + \frac{l(l+1)}{r^2} + m_i^2(r)\right] f_i(x) = \frac{E_{i,l}^2}{g_2^2(E/E_P)} f_i(x) \quad (i = 1, 2), \quad (10)$$

where we have used reduced fields of the form $f_i(x) = F_i(x)/r$, and have defined, for simplicity, two r -dependent effective masses $m_1^2(r)$ and $m_2^2(r)$ given by

$$\begin{cases} m_1^2(r) = \frac{6}{r^2} \left(1 - \frac{b(r)}{r}\right) - \frac{3b'(r)}{2r^2} + \frac{3b(r)}{2r^3} \\ m_2^2(r) = \frac{6}{r^2} \left(1 - \frac{b(r)}{r}\right) - \frac{b'(r)}{2r^2} - \frac{3b(r)}{2r^3} \end{cases} \quad (r \equiv r(x)). \quad (11)$$

Taking into account the WKB approximation, from Eq. (10) we extract two r -dependent radial wave numbers given by

$$k_i^2(r, l, \omega_{i,nl}) = \frac{E_{i,nl}^2}{g_2^2(E/E_P)} - \frac{l(l+1)}{r^2} - m_i^2(r) \quad (i = 1, 2). \quad (12)$$

It is useful to use the WKB method implemented by 't Hooft in the brick wall problem [19], by counting the number of modes with frequency less than ω_i , $i = 1, 2$. This is given by

$$\tilde{g}(E_i) = \int_0^{l_{\max}} \nu_i(l, E_i) (2l+1) dl, \quad (13)$$

where $\nu_i(l, E_i)$, $i = 1, 2$, is the number of nodes in the mode with (l, E_i) , such that $(r \equiv r(x))$

$$\nu_i(l, E_i) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dx \sqrt{k_i^2(r, l, E_i)}. \quad (14)$$

Note that the integration with respect to x and l_{\max} is taken over those values which satisfy $k_i^2(r, l, E_i) \geq 0$, $i = 1, 2$. With the help of Eqs. (13) and (14), the self-sustained traversable wormhole equation becomes

$$H_\Sigma^{(0)} = -\frac{1}{\pi} \sum_{i=1}^2 \int_0^{+\infty} E_i \frac{g_1(E/E_P)}{g_2^2(E/E_P)} \frac{d\tilde{g}(E_i)}{dE_i} dE_i. \quad (15)$$

The explicit evaluation of the density of states yields

$$\begin{aligned} \frac{d\tilde{g}(E_i)}{dE_i} &= \int \frac{\partial \nu(l, E_i)}{\partial E_i} (2l+1) dl \\ &= \frac{1}{\pi} \int_{-\infty}^{+\infty} dx \int_0^{l_{\max}} \frac{(2l+1)}{\sqrt{k_i^2(r, l, E)}} \frac{d}{dE_i} \left(\frac{E_i^2}{g_2^2(E/E_P)} - m_i^2(r) \right) dl \\ &= \frac{2}{\pi} \int_{-\infty}^{+\infty} dx r^2 \frac{d}{dE_i} \left(\frac{E_i^2}{g_2^2(E/E_P)} - m_i^2(r) \right) \sqrt{\frac{E_i^2}{g_2^2(E/E_P)} - m_i^2(r)} \\ &= \frac{4}{3\pi} \int_{-\infty}^{+\infty} dx r^2 \frac{d}{dE_i} \left(\frac{E_i^2}{g_2^2(E/E_P)} - m_i^2(r) \right)^{\frac{3}{2}}. \end{aligned} \quad (16)$$

Finally, plugging expression (16) into Eq. (15) and taking into account the energy density, we obtain the following self-sustained equation

$$\frac{1}{2G} \frac{b'(r)}{r^2 g_2(E/E_P)} = \frac{2}{3\pi^2} (I_1 + I_2), \quad (17)$$

which will play an important role in the analysis below. The integrals I_1 and I_2 are defined as

$$I_1 = \int_{E^*}^{\infty} E \frac{g_1(E/E_P)}{g_2^2(E/E_P)} \frac{d}{dE} \left(\frac{E^2}{g_2^2(E/E_P)} - m_1^2(r) \right)^{\frac{3}{2}} dE, \quad (18)$$

and

$$I_2 = \int_{E^*}^{\infty} E \frac{g_1(E/E_P)}{g_2^2(E/E_P)} \frac{d}{dE} \left(\frac{E^2}{g_2^2(E/E_P)} - m_2^2(r) \right)^{\frac{3}{2}} dE, \quad (19)$$

respectively. Note that E^* is the value which annihilates the argument of the root.

In I_1 and I_2 we have included an additional 4π factor coming from the angular integration and we have assumed that the effective mass does not depend on the energy E . To further proceed, we can see what happens to expression (17) for some specific forms of $g_1(E/E_P)$ and $g_2(E/E_P)$. It is immediate to see that integrals I_1 and I_2 can be solved when $g_2(E/E_P) = g_1(E/E_P)$. However the classical term keeps a dependence on the function $g_2(E/E_P)$ that cannot be eliminated except for the simple case of $g_2(E/E_P) = 1$. Therefore we will consider different models regulated by the Rainbow's function $g_1(E/E_P)$, of the form given by Eq. (2), to analyse the effect on the form of the shape function $b(r)$.

III. EXAMPLES

$$\textbf{A. Specific case: } g_1(E/E_P) = \exp(-\alpha \frac{E^2}{E_P^2}), \quad g_2(E/E_P) = 1$$

Following Ref. [3], we consider the following choice for the Rainbow's functions

$$g_1(E/E_P) = \exp(-\alpha \frac{E^2}{E_P^2}), \quad g_2(E/E_P) = 1; \quad \alpha > 0 \in \mathbb{R}. \quad (20)$$

Thus, the graviton contribution terms (18) and (19) yield the following relationships

$$I_1 = 3 \int_{\sqrt{m_1^2(r)}}^{\infty} \exp(-\alpha \frac{E^2}{E_P^2}) E^2 \sqrt{E^2 - m_1^2(r)} dE, \quad (21)$$

and

$$I_2 = 3 \int_{\sqrt{m_2^2(r)}}^{\infty} \exp(-\alpha \frac{E^2}{E_P^2}) E^2 \sqrt{E^2 - m_2^2(r)} dE, \quad (22)$$

respectively. Using the general results outlined in Appendix A for the two integrals I_1 and I_2 , Eq. (17) can be rearranged in the following way

$$\frac{1}{2G} \frac{b'(r)}{r^2} = \frac{E_P^4}{2\pi^2} \left[\frac{x_1^2}{\alpha} \exp\left(-\frac{\alpha x_1^2}{2}\right) K_1\left(\frac{\alpha x_1^2}{2}\right) + \frac{x_2^2}{\alpha} \exp\left(\frac{\alpha x_2^2}{2}\right) K_1\left(\frac{\alpha x_2^2}{2}\right) \right], \quad (23)$$

where $x_1 = \sqrt{m_1^2(r)/E_P^2}$, $x_2 = \sqrt{m_2^2(r)/E_P^2}$ and $K_1(x)$ is a modified Bessel function of order 1. Note that it is extremely difficult to extract any useful information from this relationship, so that in the following we consider two regimes, namely the cis-planckian regime, where $x_i \ll 1$ ($i = 1, 2$), and the trans-planckian regime, where $x_i \gg 1$.

In the cis-planckian regime, with the approximation $x_1 \ll 1$ and $x_2 \ll 1$, and expanding the right hand side of Eq. (23), we find that the leading term is given by

$$\frac{1}{2G} \frac{b'(r)}{r^2} \simeq \frac{E_P^4}{2\pi^2} \left[\frac{4}{\alpha^2} - \frac{1}{\alpha} (x_1^2 + x_2^2) + O(x_1^4 + x_2^4) \right]. \quad (24)$$

Substituting the factors $x_i = \sqrt{m_i^2(r)/E_P^2}$ ($i = 1, 2$) in the latter, provides the following relationship

$$\frac{1}{2G} \frac{b'(r)}{r^2} = \frac{E_P^4}{2\pi^2} \left[\frac{4}{\alpha^2} - \frac{12}{\alpha r^2 E_P^2} \left(1 - \frac{b(r)}{r} \right) + \frac{2b'(r)}{\alpha E_P^2 r^2} \right], \quad (25)$$

which can be rearranged to give

$$b'(r) = \frac{1}{\pi^2} \left[\frac{4r^2}{\alpha^2 G} - \frac{12}{\alpha} \left(1 - \frac{b(r)}{r} \right) + \frac{2b'(r)}{\alpha} \right], \quad (26)$$

where we have used the definition $G = E_P^{-2} = l_P^2$. Restricting our attention to the dominant term only, we find that

$$b(r) = r_t + \frac{E_P^2}{3\pi^2 \alpha^2} (r^3 - r_t^3), \quad (27)$$

which does not represent an asymptotically flat wormhole geometry, as the condition $b(r)/r \rightarrow 0$ when $r \rightarrow +\infty$, is not satisfied. However, for these cases, one may in principle match these interior wormhole solutions with an exterior vacuum Schwarzschild spacetime. We shall not proceed with the matching analysis in this paper, but we refer the reader to [15] for further details.

On the other hand, in the trans-planckian regime, i.e., $x_1 \gg 1$ and $x_2 \gg 1$, we obtain the following approximation

$$\frac{1}{2G} \frac{b'(r)}{r^2} \simeq \frac{E_P^4}{8\sqrt{\alpha^3 \pi^3}} \left[\exp(-\alpha x_1^2) x_1 + O\left(\frac{1}{x_1}\right) + \exp(-\alpha x_2^2) x_2 + O\left(\frac{1}{x_2}\right) \right]. \quad (28)$$

Note that in this regime, the asymptotic expansion is dominated by the Gaussian exponential so that the quantum correction vanishes. Thus, the only solution is $b'(r) = 0$ and consequently we have a constant shape function, namely, $b(r) = r_t$. In summary, in the trans-planckian regime provided by the Rainbow's function (20), we verify that the self-sustained equation (17) permits asymptotically flat wormhole solutions with a constant shape function given by $b(r) = r_t$.

$$\textbf{B. Specific case: } g_1(E/E_P) = \left(1 + \beta \frac{E}{E_P}\right) \exp(-\alpha \frac{E^2}{E_P^2}), \quad g_2(E/E_P) = 1$$

Another interesting choice for the Rainbow's functions is the following [3],

$$g_1(E/E_P) = \left(1 + \beta \frac{E}{E_P}\right) \exp(-\alpha \frac{E^2}{E_P^2}), \quad g_2(E/E_P) = 1; \quad \alpha > 0, \beta \in \mathbb{R}. \quad (29)$$

For these specific functions, we once again use the general results in Appendix A so that the two integrals I_1 and I_2 , given by Eqs. (18)-(19), take the following form

$$I_{1,2} = \frac{E_P^4}{2\pi^2} \left[\frac{x_{1,2}^2}{\alpha} \exp\left(-\frac{\alpha x_{1,2}^2}{2}\right) K_1\left(\frac{\alpha x_{1,2}^2}{2}\right) + \beta \frac{\sqrt{\pi}}{\alpha^{3/2}} \left(x_{1,2}^2 + \frac{3}{2\alpha}\right) \exp(-\alpha x_{1,2}^2) \right]. \quad (30)$$

where once again, we have defined for notational simplicity $x_{1,2} = \sqrt{m_{1,2}^2(r)/E_P^2}$. As in the previous example, it is extremely difficult to extract any useful information from these relationships, so that we consider the two regimes, i.e., the cis-planckian regime, where $x_i \ll 1$ ($i = 1, 2$), and the trans-planckian regime, where $x_i \gg 1$, respectively.

In the cis-planckian regime, where $x_i \ll 1$, the self-sustained equation (17) takes the following form

$$\frac{1}{2G} \frac{b'(r)}{r^2} = \frac{E_P^4}{2\alpha\pi^2} \left[\frac{2}{\alpha} + \frac{3\sqrt{\pi}\beta}{2\alpha^{3/2}} - \frac{2\sqrt{\alpha} + \sqrt{\pi}\beta}{2\alpha^{1/2}} (x_1^2 + x_2^2) + O(x^4) \right], \quad (31)$$

which leads to

$$b'(r) = \frac{E_P^2}{\alpha\pi^2} \left[\frac{4\sqrt{\alpha} + 3\sqrt{\pi}\beta}{2\alpha^{3/2}} r^2 - \frac{2\sqrt{\alpha} + \sqrt{\pi}\beta}{\alpha^{1/2}} \left(\frac{6}{E_P^2} \left(1 - \frac{b(r)}{r} \right) - \frac{b'(r)}{E_P^2} \right) \right]. \quad (32)$$

Note that the term proportional to r^2 leads to a non-asymptotically flat wormhole spacetime, so that it is useful to do away with this term. To this effect, it is straightforward to see that for $\beta = -\frac{4}{3}\sqrt{\alpha/\pi}$, one arrives at

$$b'(r) = -\frac{2}{3\alpha\pi^2} \left[6 \left(1 - \frac{b(r)}{r} \right) - b'(r) \right], \quad (33)$$

which provides the following solution

$$b(r) = \frac{1}{3\alpha\pi^2 - 12\pi^2 - 2} \left[(3\alpha\pi^2 - 2) r_t \left(\frac{r}{r_t} \right)^{\frac{12\pi^2}{3\alpha\pi^2 - 2}} - 12\pi^2 r \right], \quad (34)$$

where we have used the condition $b(r_t) = r_t$. It is a simple matter to verify that this solution is not asymptotically flat, however, as in the previous example one may match the interior solution to an exterior solution (once again, we refer the reader to [15] for further details).

On the other hand, for the trans-planckian regime, $x_i \gg 1$, the asymptotic series becomes

$$\frac{1}{2G} \frac{b'(r)}{r^2} \simeq \frac{E_P^4}{8\sqrt{\alpha^3\pi^3}} \left[(x_1 + \beta x_1^2) \exp(-\alpha x_1^2) + O\left(\frac{1}{x_1}\right) + (x_2 + \beta x_2^2) \exp(-\alpha x_2^2) + O\left(\frac{1}{x_2}\right) \right]. \quad (35)$$

Once again, we find that the quantum corrections are exponentially suppressed, and as in the previous example leads to a constant shape function, $b(r) = r_t$.

$$\mathbf{C. \quad Specific \ case:} \quad g_1(E/E_P) = \left(1 + \beta \frac{E}{E_P} + \gamma \frac{E^2}{E_P^2}\right) \exp(-\alpha \frac{E^2}{E_P^2}), \quad g_2(E/E_P) = 1$$

Finally, consider now the following choices for the Rainbow's functions given by

$$g_1(E/E_P) = \left(1 + \beta \frac{E}{E_P} + \gamma \frac{E^2}{E_P^2}\right) \exp(-\alpha \frac{E^2}{E_P^2}), \quad g_2(E/E_P) = 1. \quad (36)$$

Once again using the general results outlined in Appendix A, we find that the two integrals I_1 and I_2 , given by Eqs. (18)-(19), take the following form

$$I_{1,2} = \frac{E_P^4}{2\pi^2\alpha} \exp\left(-\frac{\alpha x_{1,2}^2}{2}\right) \left[x_{1,2}^2 K_1\left(\frac{\alpha x_{1,2}^2}{2}\right) + \frac{\beta\sqrt{\pi}}{4\sqrt{\alpha}} \exp\left(-\frac{\alpha x_{1,2}^2}{2}\right) \left(x_{1,2}^2 + \frac{3}{2\alpha}\right) \right. \\ \left. + \gamma x_{1,2}^2 \left(-\frac{x_{1,2}^2}{2\alpha} K_1\left(\frac{\alpha x_{1,2}^2}{2}\right) + \frac{x_{1,2}^2}{2} \left(-K_0\left(\frac{\alpha x_{1,2}^2}{2}\right) - \frac{2}{\alpha} K_1\left(\frac{\alpha x_{1,2}^2}{2}\right)\right) - \frac{1}{\alpha} K_1\left(\frac{\alpha x_{1,2}^2}{2}\right)\right) \right], \quad (37)$$

with the definition $x_{1,2} = \sqrt{m_{1,2}^2(r)/E_P^2}$. We now investigate the two limiting approximations, namely, the cis-planckian regime, where $x_i \ll 1$ ($i = 1, 2$), and the trans-planckian regime, where $x_i \gg 1$, respectively.

Regarding the cis-planckian regime, the self-sustained equation (17)

$$\frac{1}{2G} \frac{b'(r)}{r^2} = \frac{E_P^4}{2\alpha\pi^2} \left[2 \left(\frac{2}{\alpha^2} + \frac{3\beta\sqrt{\pi}}{2\alpha^{5/2}} - \frac{4\gamma}{\alpha^3} \right) + \left(\frac{\gamma}{\alpha^2} - \frac{\beta\sqrt{\pi}}{2\alpha^{3/2}} - \frac{1}{\alpha} \right) (x_1^2 + x_2^2) \right. \\ \left. + \left(\frac{2\ln(\alpha x_1^2/4) + 2\ln(\alpha x_2^2/4) + 2\gamma_E + 1}{8} + \frac{\gamma}{4\alpha} - \frac{\sqrt{\pi}\beta}{16\sqrt{\alpha}} \right) (x_1^4 + x_2^4) \right] + O(x^6), \quad (38)$$

where γ_E is the Euler's constant. In order to simplify the analysis, consider the following imposition

$$\begin{cases} \frac{2}{\alpha^2} + \frac{3\beta\sqrt{\pi}}{2\alpha^{5/2}} - \frac{4\gamma}{\alpha^3} = 0 \\ \frac{\gamma}{\alpha^2} - \frac{\beta\sqrt{\pi}}{2\alpha^{3/2}} - \frac{1}{\alpha} = 0 \end{cases}, \quad (39)$$

which leads to the following solution

$$\beta = -4\sqrt{\frac{\alpha}{\pi}}; \quad \gamma = -\alpha; \quad \text{for } \alpha \in \mathbb{R}^+. \quad (40)$$

Thus, Eq. (38) simplifies to

$$\frac{1}{2G} \frac{b'(r)}{r^2} = \frac{E_P^4}{2\alpha\pi^2} \left[\left(\frac{\ln(\alpha x_1^2/4) + \ln(\alpha x_2^2/4) + 2\gamma_E + 1}{4} \right) (x_1^4 + x_2^4) \right]. \quad (41)$$

Note that as the previous formula is obtained in the cis-planckian regime, the logarithmic functions in this range are always negative and the whole expression goes to zero from the negative side. Furthermore, as in the previous two examples, it is an easy matter to show that the solution is not asymptotically flat, and in principle can be matched to an exterior vacuum solution at a junction interface.

Let us now fix our attention on the trans-planckian regime, where $x_i \gg 1$ ($i = 1, 2$). The integrals in the Eq. (37) have the following asymptotic expansion

$$I_{1,2} \simeq \frac{E_P^4}{2\pi^2} \exp(-\alpha x_{1,2}^2) \frac{\sqrt{\pi}}{\alpha^{3/2}} \left(-\gamma x_{1,2}^3 + \frac{\beta x_{1,2}^2}{4} + \frac{(4\alpha - 9\gamma) x_{1,2}}{4\alpha} + \frac{3\beta}{8\alpha} + \frac{3(8\alpha - 15\gamma)}{32\alpha^2 x_{1,2}} - \frac{15(4\alpha - 7\gamma)}{128\alpha^3 x_{1,2}^3} + O(x_{1,2}^{-5}) \right). \quad (42)$$

Analogously to the previous cases, due to the Gaussian exponential, we find that quantum fluctuations lead to a vanishing contribution, independently of the choice of the parameters β and γ . Therefore we conclude that $b'(r) = 0$ leading to a constant shape function, $b(r) = r_t$.

Collecting the results of cases described in sections (III A, III B) and (III C), we conclude that the model governed by the Gaussian Rainbow's function $g_1(E/E_P)$ and its polynomial variations lead only to a constant shape function $b(r_t) = r_t$. In the next section, we consider a different approach to the problem of the self sustained equation by fixing the geometry of the wormhole. More specifically, we consider a variational approach which imposes a local analysis to the problem and restrict our attention to the behavior of the metric function $b(r)$ at the wormhole throat, r_t .

IV. FIXING THE GEOMETRY

In this section, we outline a different approach in the context of self-sustained wormholes in MDR. More specifically, we fix the shape function, and therefore the geometry, and restricting our analysis to the throat we find conditions on the specific parameter space in order to have wormhole solutions. In the semi-classical context, we emphasize that solutions of self-sustained wormholes were found in Refs. [13, 14], by using standard regularization and renormalization techniques. Thus, we apply an analogous approach, however without taking into account any renormalization procedure, but using only the deformed spacetime at the Planck scale.

To set the stage and for concreteness consider the specific choice for the shape function $b(r) = r_t^2/r$ [7]. Thus, from Eq. (17), and restricting the analysis to the throat $r = r_t$, we find

$$-\frac{1}{2G} \frac{1}{r_t^2 g_2(E)} = \frac{2}{3\pi^2} (I_1 + I_2), \quad (43)$$

with I_1 and I_2 given by Eqs. (18) and (19), respectively. Maintaining the setting $g_2(E/E_P) = 1$, we find that the effective masses at the throat simplify to

$$\begin{cases} m_1^2(r_t) = \frac{3}{r_t^2} \\ m_2^2(r_t) = -\frac{1}{r_t^2} \end{cases}. \quad (44)$$

Therefore I_1 and I_2 become

$$I_1 = 3 \int_{3/r_t^2}^{\infty} g_1(E/E_P) E^2 \sqrt{E^2 - \frac{3}{r_t^2}} dE \quad (45)$$

and

$$I_2 = 3 \int_0^{\infty} g_1(E/E_P) E^2 \sqrt{E^2 + \frac{1}{r_t^2}} dE, \quad (46)$$

respectively.

Now, in order to have only one solution with variables α and r_t , we demand that

$$\frac{d}{dr_t} \left[-\frac{1}{2G} \frac{1}{r_t^2} \right] = \frac{d}{dr_t} \left[\frac{2}{3\pi^2} (I_1 + I_2) \right], \quad (47)$$

which takes the following form

$$1 = \frac{2}{\pi^2} \left[3 \int_{\sqrt{3}/r_t E_P}^{\infty} g_1(u) u^2 \sqrt{u^2 - \frac{3}{(r_t E_P)^2}} du - \int_0^{\infty} g_1(u) u^2 \sqrt{u^2 + \frac{1}{(r_t E_P)^2}} du \right], \quad (48)$$

where we have set $u = E/E_P$ and $G^{-1} = E_P^2$.

Consider a specific form of $g_1(u)$ given by $g_1(u) = \exp(-\alpha u^2)$ with α variable, and after integration, we find that

$$1 = \frac{2}{\pi^2} \frac{d}{d\alpha} \left[\frac{1}{2} \exp\left(\frac{\alpha}{2(r_t E_P)^2}\right) K_0\left(\frac{\alpha}{2(r_t E_P)^2}\right) \right] - \frac{6}{\pi^2} \frac{d}{d\alpha} \left[\frac{1}{2} \exp\left(-\frac{3\alpha}{2(r_t E_P)^2}\right) K_0\left(\frac{3\alpha}{2(r_t E_P)^2}\right) \right], \quad (49)$$

where $K_0(x)$ is a modified Bessel function of order 0. Writing the explicit form of the derivatives in Eq. (49) we find

$$1 = \frac{1}{2\pi^2 x^2} f(\alpha, x), \quad (50)$$

where, for notational simplicity, we have used $x = r_t E_P$ and used the definition

$$f(\alpha, x) = \exp\left(\frac{\alpha}{2x^2}\right) K_0\left(\frac{\alpha}{2x^2}\right) - \exp\left(\frac{\alpha}{2x^2}\right) K_1\left(\frac{\alpha}{2x^2}\right) + 9 \exp\left(-\frac{3\alpha}{2x^2}\right) K_0\left(\frac{3\alpha}{2x^2}\right) + \exp\left(-\frac{3\alpha}{2x^2}\right) K_1\left(\frac{3\alpha}{2x^2}\right). \quad (51)$$

The procedure now is in principle straightforward. In order to have one and only one solution, we demand that the expression in the r.h.s. of Eq. (50) has a stationary point with respect to x which coincides with the constant value 1. For a generic but small α , we can expand in powers of α to find

$$0 = \frac{d}{dx} \left[\frac{1}{2\pi^2 x^2} f(\alpha, x) \right] \simeq \frac{20 - 10 \ln(4x^2/\alpha) + 10\gamma_E + 9 \ln 3}{\pi^2 x^2} + O(\alpha), \quad (52)$$

which has a root at

$$\bar{x} = r_t E_P = 2.973786871 \sqrt{\alpha}. \quad (53)$$

Substituting \bar{x} into Eq. (50), we find

$$1 = \frac{0.2423530631}{\alpha}, \quad (54)$$

fixing therefore $\alpha \simeq 0.242$. It is interesting to note that this value is very close to the value $\alpha = 1/4$ used in Ref. [3] inspired by a non-commutative analysis [20]. As in Ref. [14], it is rather important to emphasize a shortcoming in the analysis carried in this section, mainly due to the technical difficulties encountered. Note that we have considered a variational approach which imposes a local analysis to the problem, namely, we have restricted our attention to the behavior of the metric function $b(r)$ at the wormhole throat, r_t . Despite the fact that the behavior is unknown far from the throat, due to the high curvature effects at or near r_t , the analysis carried out in this section should extend to the immediate neighborhood of the wormhole throat. Nevertheless it is interesting to observe that in Ref. [13] the greatest value of the wormhole throat was fixed at $r_t \simeq 1.16/E_P$ using a regularization-renormalization scheme. From Eq. (53), one immediately extracts $r_t \simeq 1.46/E_P$ which is slightly larger.

V. SUMMARY AND DISCUSSION

Wormhole are hypothetical tunnels that violate the null energy condition, and therefore all of the energy conditions, and thus it seems that a natural environment of these exotic spacetimes lies in the quantum regime, as a large number of quantum systems have been shown to violate the energy conditions [7, 8], such as the Casimir effect. In this context, it has been shown that various wormhole solutions in semi-classical gravity have been found in the literature [12–14]. In the semi-classical approach, the Einstein field equation takes the form $G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle^{\text{ren}}$, where the term $\langle T_{\mu\nu} \rangle^{\text{ren}}$ is the renormalized expectation value of the stress-energy tensor operator of the quantized field. In addition to this, the metric is separated into a background component, $\bar{g}_{\mu\nu}$ and a perturbation $h_{\mu\nu}$, i.e., $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. A key aspect is that the Einstein tensor may also be separated into a part describing the curvature due to the background geometry and that due to the perturbation, namely, $G_{\mu\nu}(g_{\alpha\beta}) = G_{\mu\nu}(\bar{g}_{\alpha\beta}) + \Delta G_{\mu\nu}(\bar{g}_{\alpha\beta}, h_{\alpha\beta})$ where $\Delta G_{\mu\nu}(\bar{g}_{\alpha\beta}, h_{\alpha\beta})$ may be considered a perturbation series in terms of $h_{\mu\nu}$. Using the semi-classical Einstein field

equation, in the absence of matter fields, the effective stress-energy tensor for the quantum fluctuations is given by $8\pi G \langle T_{\mu\nu} \rangle^{\text{ren}} = -\langle \Delta G_{\mu\nu}(\bar{g}_{\alpha\beta}) \rangle^{\text{ren}}$ so that the equation governing the quantum fluctuations behaves as a backreaction equation. Thus, the possibility that wormhole geometries were sustained by their own quantum fluctuations, using the above-mentioned semi-classical approach was shown in [13, 14]. More specifically, the graviton one loop contribution to a classical energy in a wormhole background was taken into account, and a variational approach with Gaussian trial wave functionals was used. A zeta function regularization was involved to handle with divergences and a renormalization procedure was introduced and the finite one loop energy was considered as a *self-consistent* source for the traversable wormhole.

In this work, we have considered the possibility that wormhole geometries are also sustained by their own quantum fluctuations, but in the context of modified dispersion relations. We considered different models regulated by the Rainbow's function $g_1(E/E_P)$, given by Eq. (2), to analyse the effect on the form of the shape function $b(r)$, and found specific solutions for wormhole geometries in the cis-planckian regime and trans-planckian regime. In the latter regime, we found that the quantum correction are exponentially suppressed, thus leading to a constant shape function, i.e., $b(r) = r_t$, where r_t is the wormhole throat. In the cis-planckian regime, the solutions found do not represent asymptotically flat wormhole geometries. However, for these cases, one may in principle match these interior wormhole solutions with an exterior vacuum Schwarzschild spacetime. In addition to this analysis, we also fixed the geometry by considering the behaviour of a specific shape function $b(r)$, through a variational approach, at the wormhole throat. We explored the parameter range of the Rainbow's functions and found a good agreement with previous results used in the literature [3]. It is important to point out that the above-mentioned variational approach presents a shortcoming mainly due to the technical difficulties encountered. Note that the latter variational approach considered imposes a local analysis to the problem, namely, we have restricted our attention to the behavior of the metric function $b(r)$ at the wormhole throat, r_t (this is similar to the analysis carried out in [14]). Despite the fact that the behavior is unknown far from the throat, due to the high curvature effects at or near r_t , the analysis carried out in this context should extend to the immediate neighborhood of the wormhole throat.

Appendix A: Integrals

In this Appendix we provide the rules to solve the integrals I_1 and I_2 , given by Eqs. (18)-(19), with the following choices for the Rainbow's functions

$$g_1\left(\frac{E}{E_P}\right) = \sum_{i=0}^n \beta_i \frac{E^i}{E_P^i} \exp\left(-\alpha \frac{E^2}{E_P^2}\right), \quad g_2(E/E_P) = 1; \quad \alpha > 0, \beta_i \in \mathbb{R}. \quad (\text{A1})$$

In some particular cases such as in section IV, the effective mass becomes negative and the integral takes the form

$$I_+ = \int_0^\infty g_1\left(\frac{E}{E_P}\right) E^2 \sqrt{E^2 + m^2} dE. \quad (\text{A2})$$

Therefore we divide the integration in two classes denoted by the integrals I_+ and I_- described by

$$I_- = \int_{\sqrt{m^2}}^\infty g_1\left(\frac{E}{E_P}\right) E^2 \sqrt{E^2 - m^2} dE. \quad (\text{A3})$$

For the examples discussed in section III, the following relationships for I_- are extremely useful

$$\frac{d^n I_-}{d\alpha^n} = (-)^n E_P^{2n} \int_{\sqrt{m^2}}^\infty \exp\left(-\alpha \frac{E^2}{E_P^2}\right) E^{2n} \sqrt{E^2 - m^2} dE \quad n \geq 1 \quad (\text{A4})$$

and the following for I_+

$$\frac{d^n I_+}{d\alpha^n} = (-)^n E_P^{2n} \int_0^\infty \exp\left(-\alpha \frac{E^2}{E_P^2}\right) E^{2n} \sqrt{E^2 + m^2} dE \quad n \geq 1. \quad (\text{A5})$$

Taking into account $g_1\left(\frac{E}{E_P}\right) = \exp\left(-\alpha \frac{E^2}{E_P^2}\right)$ in I_- , we find that changing variables $E = \sqrt{x}$, we obtain

$$\begin{aligned} I_- &= \frac{1}{2} \int_{\sqrt{m^2}}^\infty \exp\left(-\alpha \frac{x}{E_P^2}\right) \sqrt{x} \sqrt{x - m^2} dx \\ &= \frac{E_P^4}{2\sqrt{\pi}} \left(\frac{m^2}{\alpha E_P^2}\right) \Gamma\left(\frac{3}{2}\right) \exp\left(-\frac{\alpha m^2}{2E_P^2}\right) K_1\left(\frac{\alpha m^2}{2E_P^2}\right), \end{aligned} \quad (\text{A6})$$

where we have used the following relationship

$$\int_u^\infty (x-u)^{\mu-1} x^{\mu-1} \exp(-\beta x) dx = \frac{1}{\sqrt{\pi}} \left(\frac{u}{\beta}\right)^{\mu-1/2} \Gamma(\mu) \exp\left(-\frac{\beta u}{2}\right) K_{\mu-1/2}\left(\frac{\beta u}{2}\right) \quad \begin{matrix} \text{Re } \mu > 0 \\ \text{Re } \beta u > 0 \end{matrix} . \quad (\text{A7})$$

The same argument applies for I_+ to obtain

$$\begin{aligned} I_+ &= \frac{1}{2} \int_0^\infty \exp(-\alpha \frac{x}{E_P^2}) \sqrt{x} \sqrt{x+m^2} dx \\ &= \frac{E_P^4}{2\sqrt{\pi}} \left(\frac{m^2}{\alpha E_P^2}\right) \Gamma\left(\frac{3}{2}\right) \exp\left(\frac{\alpha m^2}{2E_P^2}\right) K_{-1}\left(\frac{\alpha m^2}{2E_P^2}\right), \end{aligned} \quad (\text{A8})$$

where we have used the following relationship

$$\int_0^\infty (x+\beta)^{\nu-1} x^{\nu-1} \exp(-\mu x) dx = \frac{1}{\sqrt{\pi}} \left(\frac{\beta}{\mu}\right)^{\nu-1/2} \Gamma(\nu) \exp\left(\frac{\beta \mu}{2}\right) K_{1/2-\nu}\left(\frac{\beta \mu}{2}\right) \quad \begin{matrix} \text{Re } \mu > 0 \\ \text{Re } \nu > 0 \\ |\arg \beta| < \pi \end{matrix} . \quad (\text{A9})$$

On the other hand, the integrals of the form

$$\frac{d^n I_-^o}{d\alpha^n} = (-)^n E_P^{2n} \int_{\sqrt{m^2}}^\infty \exp(-\alpha \frac{E^2}{E_P^2}) E^{2n+1} \sqrt{E^2 - m^2} dE \quad n \geq 2, \quad (\text{A10})$$

can be generated by the following expression

$$\int_a^\infty dx (x-a)^{1/2} x \exp(-\mu x) = \frac{\sqrt{\pi}}{4} \mu^{-5/2} (3+2\mu a) \exp(-\mu a) \quad a > 0, \mu > 0, \quad (\text{A11})$$

while integrals of the form

$$\frac{d^n I_+^o}{d\alpha^n} = (-)^n E_P^{2n} \int_0^\infty \exp(-\alpha \frac{E^2}{E_P^2}) E^{2n+1} \sqrt{E^2 + m^2} dE \quad n \geq 2 \quad (\text{A12})$$

can be generated by the following relationship

$$\int_0^\infty dx (x+t)^{1/2} x \exp(-\mu x) = \frac{3}{2} \frac{\sqrt{t}}{\mu^2} + \frac{\sqrt{\pi}}{4} \mu^{-5/2} \exp(t\mu) (3-2t\mu) \text{Erfc}[\sqrt{t\mu}] \quad t > 0, \mu > 0. \quad (\text{A13})$$

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